FANO HYPERSURFACES WITH NO FINITE ORDER BIRATIONAL AUTOMORPHISMS

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ABSTRACT. We use the specialization homomorphism for the birational automorphism group to study finite order birational automorphisms. For a family of varieties over a DVR, we prove that a birational automorphism of order coprime to the residue characteristic cannot specialize to the identity. As an application, we show that very general n-dimensional hypersurfaces of degree $d \geq 5\lceil (n+3)/6 \rceil$ have no finite order birational automorphisms.

The birational automorphism group of a variety X—denoted $\mathrm{Bir}(X)$—is one of the most natural birational invariants associated to X. For $X = \mathbb{P}^n_{\mathbb{C}}$, the *Cremona group* $\mathrm{Cr}_n(\mathbb{C}) = \mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$ is an object of classical and modern interest, and it is extremely interesting and complicated when $n \geq 2$. Beyond the case of projective space, it is natural to study the birational automorphism group of a smooth degree d hypersurface $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$. In the general type case, if $n \geq 2$ and $d \geq n+3$, then K_X is ample and Matsumura [15] showed that $\mathrm{Bir}(X)$ is equal to the automorphism group $\mathrm{Aut}(X)$. If $d = n+2$, in which case X is Calabi–Yau (with Picard rank 1 if $n \geq 3$), then again $\mathrm{Bir}(X) = \mathrm{Aut}(X)$ [17] (see also [13, Lem A.1]). However, if $d \leq n+1$, in which case X is Fano, very little is known about birational automorphisms in general once $n \geq 4$.

The most striking known result is the case of degree $d = n+1$ Fano hypersurfaces. To briefly summarize, there has been a great deal of work by many authors—including Fano, Segre, Iskovskikh, Manin, Pukhlikov, Corti, Cheltsov, de Fernex, Ein, Mustaţă, and Zhuang—to show that if $n \geq 3$ and $d = n+1$, then any such smooth X is birationally superrigid. As a consequence of their work, $\mathrm{Bir}(X) = \mathrm{Aut}(X)$ (see [11] for a survey of the main ideas that were developed over time). In the case $d = n$, Pukhlikov used similar techniques to show that such hypersurfaces also satisfy $\mathrm{Bir}(X) = \mathrm{Aut}(X)$ once $n \geq 14$ [20, Cor. 1]. For a smooth hypersurface X, having $\mathrm{Bir}(X) = \mathrm{Aut}(X)$ places strong constraints on the groups, as shown by Matsumura and Monsky [16, Thm. 2 and Thm. 5]: (1) if $n \geq 2$ and $d \geq 3$ (excluding the case $(n,d) = (2,4)$), then $\mathrm{Aut}(X)$ is naturally identified with a finite subgroup of $\mathrm{Aut}(\mathbb{P}^{n+1}_{\mathbb{C}}) = \mathrm{PGL}_{n+2}(\mathbb{C})$, and (2) if $n \geq 2$, $d \geq 3$, and X is very general, then $\mathrm{Aut}(X)$ is trivial. There seem to be few known restrictions on Bir when $d < n$.

We first prove a result about specializing finite order elements in the birational automorphism group (Proposition 2.1). For a family of varieties over a complex curve, this shows that a nontrivial finite order birational automorphism cannot specialize to the identity on the central fiber. We apply our result to hypersurfaces, but we believe that this specialization method will also be useful for studying the birational automorphism groups of other varieties.

2020 *Mathematics Subject Classification.* Primary: 14E07. Secondary: 14D06, 14J70.

During the preparation of this article, the first author was partially supported by an NSF postdoctoral fellowship DMS-2103099, the second author was partially supported by NSF grant DMS-1840234, and the third author was partially supported by NSF grant DMS-1952399.

By degenerating to a reducible hypersurface, our result will imply that if a very general non-ruled degree d hypersurface has no p-torsion in its birational automorphism group, then the same also holds in degree $d+1$. By degenerating to positive characteristic—following the work of Kollár [9]—we can control the torsion in $\mathrm{Bir}(X)$ for certain hypersurfaces in the Fano range.

Theorem A. *Let p be a prime and let n and d be integers; if $p = 2$ further assume that n is even. Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a very general hypersurface. If $d \geq p\left\lceil \frac{n+3}{p+1} \right\rceil$, then any finite order element in* $\mathrm{Bir}(X)$ *has order p^r for some r.*

When $d \geq n+2$, Theorem A is well known as K_X is ample or trivial. When $d = n+1$, or when $d = n$ and $n \geq 14$, we use the known results on index one and two Fano hypersurfaces. So our contribution to Theorem A is for Fano hypersurfaces of degree $d \leq n$.

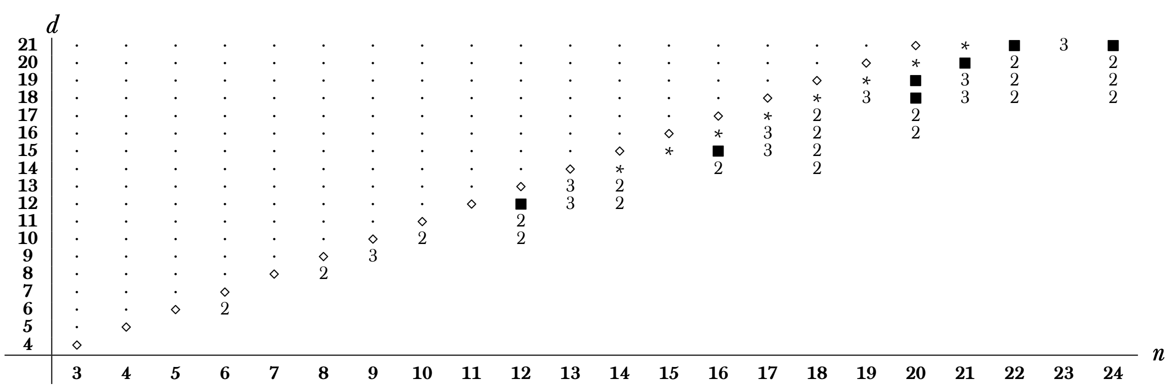
We achieve the largest range of degrees in which we can apply Theorem A by choosing the smallest primes. This gives the following corollary.

Corollary B. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a very general degree d hypersurface. If either*

(1) $d \geq 3\lceil \frac{n+3}{4} \rceil$ and n is even, or
(2) $d \geq 5\lceil \frac{n+3}{6} \rceil$ and n is odd,

Then $\mathrm{Bir}(X)$ *has no elements of finite order.*

The table below places our results in the context of previous work for some values of (n, d). The number 2 means that any finite order element in $\mathrm{Bir}(X)$ has order a power of 2 (possibly order 1) as a result of Theorem A, and similarly for 3.



■ = Corollary B · = non-Fano ◇ = Fano index 1 ⋆ = Pukhlikov's Fano index 2 results

Restricting the possible orders of torsion elements places strong restrictions on the birational automorphism group. Since the Cremona group contains p-torsion for any p, Theorem A with $p = 2$ if n is even and $p = 3$ if n is odd implies that $\mathrm{Bir}(X) \not\hookrightarrow \mathrm{Cr}_n(\mathbb{C})$ if $d \geq 2\lceil \frac{n+3}{3} \rceil$ for n even and $d \geq 3\lceil \frac{n+3}{4} \rceil$ for n odd. Remarkably, Cantat proved $\mathrm{Bir}(X) \not\hookrightarrow \mathrm{Cr}_n(\mathbb{C})$ whenever X is *any* irrational variety [2, Thm. C].

Remark 0.1. The parity assumption on n in Theorem A comes from studying the singularities of odd dimensional double covers of hypersurfaces in characteristic 2 [5, Thm. C]. At the moment, we cannot give an explicit resolution in this case.

In light of Corollary B, which shows that $\mathrm{Bir}(X)$ contains no finite order elements, one might wonder how far apart this is from showing that $\mathrm{Bir}(X) = \{1\}$. (Recall that $\mathrm{Aut}(X) = \{1\}$ for these hypersurfaces.) There are a number of related works in this vein. Any finite order element of $\mathrm{Bir}(X)$ is regularizable, i.e. it is equivalent to a regular automorphism on a birational model of X. For surfaces and for birationally rigid Fano threefolds, the regularizable automorphisms generate the birational automorphism group (e.g. $\mathrm{Cr}_2(\mathbb{C})$ is generated by $\mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}_3(\mathbb{C})$ and the Cremona involution). Cheltsov has asked whether this holds in general [3, Conj. 1.12]. Recently, Lin and Shinder [13] proved that this is false by showing that for $n \geq 3$, $\mathrm{Cr}_n(\mathbb{C})$ is not generated by (pseudo-)regularizable elements.

Throughout the paper we consider Bir as a group, not as a group scheme. However, for non-uniruled varieties Hanamura has several results on giving Bir a scheme structure [7, 8].

Notation. R will denote a DVR with field of fractions $K = \mathrm{Frac}\, R$ and residue field k. We will write η for the generic point of $\mathrm{Spec}\, R$ and 0 for the closed point.

Outline. Let X be a family over R and let $Z \subset X_0$ be a component of the special fiber. In §1, we first identify a subgroup $\Xi_\eta(Z) \subset \mathrm{Bir}_K(X_\eta)$, consisting of the birational automorphisms of the generic fiber X_η that "specialize". We construct a specialization homomorphism

$$\mathrm{sp}_\eta \colon \Xi_\eta(Z) \to \mathrm{Bir}_k(Z).$$

Next, we study torsion in the birational automorphism group in §2 and show that if ℓ is a positive integer that is invertible in R, then the kernel of the specialization map cannot contain birational automorphisms $\phi \in \Xi_\eta(Z)$ of order ℓ (see Proposition 2.1(3)). In §3 we degenerate to characteristic $p > 0$ and take advantage of some nice properties that are satisfied by the special fiber X_0 to show that $\Xi_\eta(Z)$ coincides with $\mathrm{Bir}(X_\eta)$. In particular, we use the fact (building on work of Kollár [9] and of the first and third authors [4]) that certain p-cyclic covers in characteristic p have no birational automorphisms. This is finally applied to families of hypersurfaces to prove Theorem A and Corollary B.

Acknowledgements. We are grateful to Jérémy Blanc, Michel Brion, Serge Cantat, János Kollár, Davesh Maulik, Aleksandr Pukhlikov, Evgeny Shinder, Burt Totaro, Ziquan Zhuang, and Susanna Zimmermann for helpful conversations. The first author would like to thank Professor Pietro Pirola and the University of Pavia for the opportunity to visit and their warm hospitality, during which parts of this paper were drafted.

1. The specialization homomorphism for Bir

The specialization homomorphism was first defined by Matsusaka and Mumford (who attribute it to Artin) [17], and it has also appeared in the literature for surfaces [19, §3.1] [12, §2]. To our knowledge, it has not previously been applied to systematically study birational automorphisms.

Definition 1.1 ([17, Thm I]). Let X_R be an integral flat separated scheme over R, and let $Z \subset X_0$ be a reduced irreducible component that appears with multiplicity one in the central fiber. Let $\phi \in \mathrm{Bir}_K(X_\eta)$ be a birational automorphism, and let $\Gamma \subset X_R \times_R X_R$ be the closure of the graph of ϕ. We say ϕ *specializes to* Z if the special fiber Γ_0 has a unique component that maps birationally to Z under both projections.

Example 1.2. In the ruled setting, a birational automorphism of X_η need not specialize. For the automorphism $x \mapsto \frac{t}{x}$ on the generic fiber of the constant family $\mathbb{P}^1_x \times \mathbb{A}^1_t \to \mathbb{A}^1_t$, the special fiber Γ_0 has two irreducible components, each of which is contracted under one of the projections.

Definition 1.3 ([5, Def. 1.1, Def. 1.5]). A normal scheme X has *(separably uni-)ruled modifications* if every exceptional divisor of every normal birational modification $Y \to X$ is (separably uni-)ruled. A normal scheme X_R has *sustained (separably uni-) ruled modifications* if there exists a generically finite extension of DVRs $R \subset R'$ such that for every generically finite extension of DVRs $R' \subset S$, the normalization of X_S has (separably uni-)ruled modifications. Here we fix an algebraic closure of K, and the ring extension $R \subset R'$ being generically finite means that Frac R' is a finite algebraic extension of K.

Proposition 1.4 (The specialization homomorphism). *Let X_R be an integral flat separated scheme over R and $Z \subset X_0$ a reduced irreducible component appearing with coefficient one in the special fiber.*

(1) If ϕ is a birational automorphism of X_η that specializes to Z, then there are open sets $U_1, U_2 \subset X_R$ such that each U_i meets Z, ϕ gives an isomorphism between U_1 and U_2, and the restriction of ϕ to X_0 is an isomorphism:

$$\phi|_{X_0 \cap U_1} \colon Z \cap U_1 \cong Z \cap U_2.$$

(2) The set of birational automorphisms that specialize to Z forms a subgroup of $\mathrm{Bir}_K(X_\eta)$, which we denote $\Xi_\eta(Z)$. There is a specialization group homomorphism:

$$\mathrm{sp}_\eta \colon \Xi_\eta(Z) \to \mathrm{Bir}_k(Z).$$

(3) Assume X_η and Z are geometrically integral over K and k, respectively. The group $\Xi_{\overline{\eta}}(Z_{\overline{k}})$ is the colimit of $\Xi_{\eta'}(Z_{k'})$ over generically finite extensions $R \subset R'$ of DVRs; thus, there is an induced specialization homomorphism:

$$\mathrm{sp}_{\overline{\eta}} \colon \Xi_{\overline{\eta}}(Z_{\overline{k}}) \to \mathrm{Bir}_{\overline{k}}(Z_{\overline{k}}).$$

(4) [10, IV Ex. 1.17.3] *Assume that X_R is proper and has (separably uni-)ruled modifications, and that Z is the unique irreducible component of X_0 that is not (separably uni-)ruled. Then every birational automorphism of X_η specializes to Z. That is, $\Xi_\eta(Z) = \mathrm{Bir}_K(X_\eta)$.*

(5) In the setting of (4), assume furthermore that X_R has sustained (separably uni-)ruled modifications; that X_η and Z are geometrically integral over K and k, respectively; and that $Z_{\overline{k}}$ is not (separably uni-)ruled over $\overline{k}$. Let $R \subset R''$ be a generically finite extension of DVRs. Then $\mathrm{Bir}(X_\eta)$ is a subgroup of $\mathrm{Bir}(X_{\eta''})$, and there is a further generically finite extension $R'' \subset R'$ of DVRs such that the diagram commutes:

$$\begin{array}{ccc} \mathrm{Bir}_K(X_\eta) & \xrightarrow{\mathrm{sp}_\eta} & \mathrm{Bir}_k(Z) \\ \downarrow & & \downarrow \\ \mathrm{Bir}_{K'}(X_{\eta'}) & \xrightarrow{\mathrm{sp}_{\eta'}} & \mathrm{Bir}_{k'}(Z_{k'}). \end{array}$$

In particular, there is a homomorphism

$$\mathrm{sp}_{\overline{\eta}} \colon \mathrm{Bir}_{\overline{K}}(X_{\overline{\eta}}) \to \mathrm{Bir}_{\overline{k}}(Z_{\overline{k}}).$$

Proof. Let $\tilde{\phi} \colon X_R \dashrightarrow X_R$ be the birational map over R obtained from the closure $\Gamma \subset X_R \times_R X_R$ of the graph of ϕ. Let Γ_0' be the unique component of Γ_0 mapping birationally to Z under both projections, $U_1 \subset X_R$ the largest open subset on which $\tilde{\phi}$ is an isomorphism, and $U_2 = \tilde{\phi}(U_1)$. Each U_i meets Z by maximality, and $\phi_0 = \tilde{\phi}|_Z$ as rational maps. This proves (1).

For (2), it is clear that the identity on X_η specializes to the identity on Z. If ϕ specializes to Z, then so does ϕ^{-1} by exchanging the first and second projections. It remains to show that if ϕ and ψ specialize to Z, then so does $\psi \circ \phi$, and that the specialization of the composition is the composition of the specializations. For this, let $\phi, \psi \in \Xi_\eta(Z)$, and let $\tilde{\phi}|_{U_{1,\tilde{\phi}}} \colon U_{1,\tilde{\phi}} \to U_{2,\tilde{\phi}}$ and $\tilde{\psi}|_{U_{1,\tilde{\psi}}} \colon U_{1,\tilde{\psi}} \to U_{2,\tilde{\psi}}$ be morphisms defined on the largest open subsets on which $\tilde{\phi}$ and $\tilde{\psi}$, respectively, induce isomorphisms. Let $U_2 = U_{2,\tilde{\phi}} \cap U_{1,\tilde{\psi}}, U_1 = \tilde{\phi}^{-1}(U_2)$, and $U_3 = \tilde{\psi}(U_2)$. Then each $U_i \cap Z \neq \varnothing$, so the assertion follows from the fact that $\tilde{\psi}|_{U_2\cap Z} \circ \tilde{\phi}|_{U_1\cap Z} = (\tilde{\psi} \circ \tilde{\phi})|_{U_1\cap Z}$.

For (3), if $R \subset R'$ is a generically finite extension of DVRs, then $X_{\eta'}$ and $Z' := Z \otimes_k k'$ are both integral and Z' has coefficient one in the special fiber, so they satisfy the assumptions in Definition 1.1. If Γ is the closure of the graph of an element of $\Xi_\eta(Z)$, then by assumption it has a unique component mapping birationally to Z under the projections. Therefore, the base change to k' gives a component of the special fiber of $\Gamma \otimes_R R'$ birational to $Z_{k'}$ under the projections, and there is a unique such component since $\Gamma \otimes_R R' \to R'$ is flat. This proves that $\Xi_\eta(Z)$ is a subgroup of $\Xi_{\eta'}(Z')$.

Before showing (4), first suppose that Y_R and Y'_R are flat integral schemes over Spec (R) such that Y_R has (separably uni-)ruled modifications, every non-(separably uni-)ruled component of Y_0 appears with coefficient one in Y_0, Y'_R is proper, and Y'_0 has a unique irreducible component Z' that is not (separably uni-)ruled. Then any birational map $\phi\colon Y_\eta \dashrightarrow Y'_\eta$ induces a birational map $\phi_0\colon Z \dashrightarrow Z'$ from some component $Z = Z_\phi$ of Y_0 that is not (separably uni-)ruled (c.f. [10, IV Ex. 1.17]). For this claim, first observe that the assumption on the coefficients of Y_0 implies that the local ring at the generic point of every non-(separably uni-)ruled component of Y_0 is a DVR. Now let Γ be the closure of the graph of ϕ in $Y_R \times_R Y'_R$, and let Γ'_0 be the unique component of Γ_0 mapping birationally to Z'. Since Y_R has (separably uni-)ruled modifications and Z' is not (separably uni-)ruled, then Γ'_0 maps birationally to a component Z of Y_0, so the composition $\phi_0\colon Z \dashrightarrow \Gamma'_0 \dashrightarrow Z'$ is a birational map.

We will now apply this to $X_R = Y_R = X'_R$ and $Z = Z'$ to prove (4). Let $U_1 \subset X$ be the largest open subset on which $\tilde{\phi}$ is an isomorphism, and let $U_2 = \tilde{\phi}(U_1)$. Note that $\tilde{\phi}^{-1}(U_2 \cap X_0) = U_1 \cap X_0$, so each U_i meets Z by maximality, and $\phi_0 = \tilde{\phi}|_Z$ as rational maps.

For (5), let $R \subset R'$ be as in Definition 1.3. After replacing R' by a localization of its integral closure in $K' \otimes_K K''$ we may assume $R \subset R'' \subset R'$. Then $X_{\eta'}$ and $Z' := Z_{k'}$ are integral, and Z' appears with coefficient one in the central fiber of $X_{R'}$, so the local ring of $X_{R'}$ at the generic point of Z' is a DVR. Thus, the normalization $X^\nu_{R'} \to X_{R'}$ is an isomorphism at the generic point of Z', so on the special fiber there is a component W mapping birationally to Z'. Now we apply (4) to obtain a specialization map

$$\mathrm{Bir}_{K'}(X_{\eta'}) = \mathrm{Bir}_{K'}(X^\nu_{\eta'}) \to \mathrm{Bir}_{k'}(W) = \mathrm{Bir}_{k'}(Z').$$

(5) then follows from (3). □

Let X_R be a family of smooth proper varieties. The previous proposition describes how to specialize birational automorphisms, and one may wonder what the image of the subgroup $\mathrm{Aut}_K(X_\eta)\cap\Xi_\eta(Z) \subset \mathrm{Bir}_K(X_\eta)$ is in $\mathrm{Bir}_k(X_0)$. Let $\phi \in \mathrm{Aut}_K(X_\eta) \cap \Xi_\eta(Z)$. If there is an ample divisor $\mathcal{L}$ on X_η such that $\mathcal{L}$ and $\phi^*\mathcal{L}$ both extend to relatively ample divisors on the family X_R, then a theorem of Matsusaka and Mumford shows that ϕ extends to a (regular) automorphism $\tilde{\phi} \in \mathrm{Aut}_R(X_R)$ and that $\mathrm{sp}_\eta(\phi)$ is a (regular) automorphism of X_0 [17, Cor. 1]. Without this additional assumption that ϕ preserves an ample class, one may ask:

Question 1.5. Is there a smooth proper family X_R and an element $\phi \in \mathrm{Aut}(X_\eta) \cap \Xi_\eta(X_0)$ such that $\mathrm{sp}_\eta(\phi) \in \mathrm{Bir}(X_0)$ is not a regular automorphism?

In the next section, we will give an example of a family of K3 surfaces and an element $\iota \in \mathrm{Aut}_K(X_\eta)$ which does not extend to a regular automorphism in $\mathrm{Aut}_R(X_R)$ (Example 2.4). In our example $\mathrm{Aut}(X_0) = \mathrm{Bir}(X_0)$, so $\mathrm{sp}_\eta(\iota)$ is still a regular automorphism of X_0.

2. Kernel of the specialization homomorphism

In this section, we study the kernel of the specialization homomorphism from §1. After regularizing an order ℓ birational automorphism on a birational model of X_R, our argument shows that any component of the special fiber fixed by the $\mathbb{Z}/\ell\mathbb{Z}$ group action must be a multiple component.

Proposition 2.1. *Let X_R be an integral flat separated scheme over R and $Z \subset X_0$ an irreducible component. Let $\phi \in \Xi_\eta(Z)$ be a birational automorphism of order ℓ, for some integer $\ell > 1$.*

(1) There is an affine open $U \subset X_R$ meeting Z on which ϕ induces an automorphism over R.
(2) If ℓ is invertible in R, then the quotient $U/\langle\phi\rangle$ exists and $(U/\langle\phi\rangle)_0 = (U \cap Z)/\langle \mathrm{sp}_\eta(\phi)\rangle$.
(3) If ℓ is invertible in R, then $\mathrm{sp}_\eta(\phi)$ has order ℓ in $\mathrm{Bir}_k(X_0)$. In particular, $\phi \notin \ker(\mathrm{sp}_\eta)$.

Proof. Let $\tilde{\phi} \in \mathrm{Bir}_R(X_R)$ be induced by ϕ, and set $U = \bigcap_{i=1}^{\ell-1} \tilde{\phi}^i(U')$, where $U' \subset U_1 \cap U_2$ is an affine open subset meeting Z, and U_1 and U_2 are as in Proposition 1.4(1). This shows (1).

Now let $U = \mathrm{Spec}\, A$, and let $\phi \in \mathrm{Aut}_R(A)$ denote the induced automorphism. We write $(-)^\phi$ to mean the submodule of ϕ-invariant elements of an A-module. The quotient Spec (A^ϕ) is integral and normal [6, Thm. 4.16], and it remains to show that $(A^\phi) \otimes_R k \cong (A \otimes_R k)^\phi$.

Left exactness of $(-)^\phi$ implies $A^\phi/(\pi A)^\phi \hookrightarrow (A \otimes_R k)^\phi$ is injective. Since ϕ is an automorphism over R and $R \to A$ is flat, we have $(\pi A)^\phi = \pi(A^\phi)$ and $A^\phi/(\pi A)^\phi \cong (A^\phi) \otimes_R k$, where π is a uniformizer of R. This shows injectivity of $(A^\phi) \otimes_R k \to (A \otimes_R k)^\phi$. For surjectivity, let $a \in (A \otimes_R k)^\phi$ and let $\tilde{a} \in A$ be a lift. Then $\frac{1}{\ell}\sum_{i=0}^{\ell-1} \phi^i(\tilde{a})$ is an element of A^ϕ mapping to $a \in A \otimes_R k$. This shows (2).

For (3), let $G = \langle\phi\rangle \subset \mathrm{Aut}_R(U)$. Since the quotient morphism $q\colon U \to (U/G)$ is a morphism over R, the pullback of the effective Cartier divisor $(U/G)_0$ on U/G is U_0 [22, Tag 01WV,Tag 0C4U]. The projection formula [14, Ch. 9 Prop. 2.11] yields $q_*[q^*((U/G)_0)] = \ell[(U/G)_0]$. The restriction of q to U_0 is thus a finite morphism of degree ℓ, so the order of $\mathrm{sp}_\eta(\phi)$ in $\mathrm{Bir}(X_0)$ must be ℓ. ◻

Remark 2.2. Proposition 2.1(3) shows that the kernel of the specialization homomorphism does not contain any torsion of order coprime to the characteristic of the residue field of R. In some special cases, the kernel is even trivial: when the specialization homomorphism $\mathrm{Pic}(X_{\overline{\eta}}) \to \mathrm{Pic}(X_{\overline{0}})$ is an isomorphism and $\mathrm{H}^0(X_{\overline{0}}, \mathcal{T}_{X_{\overline{0}}}) = 0$, Lieblich and Maulik show using the Matsusaka–Mumford theorem [17, Cor. 1] and a deformation theory argument that the specialization homomorphism is injective [12, §2].

However, injectivity does not hold in general. We now give a series of examples exhibiting nontrivial elements in the kernel of the specialization map.

Example 2.3. Let k be a field, and let $P_1, P_2, Q_t \in \mathbb{P}^2(k)$ be points such that P_1, P_2, Q_t are not collinear for $t \neq 0$, but P_1, P_2, Q_0 lie on a common line L. Denote the subscheme $P_1 + P_2 + Q_t$ by Y_t, and consider the linear system of conics with base locus Y_t. For $t \neq 0$ this defines the quadratic transformation with base locus P_1, P_2, Q_t, but on the special fiber $\mathbb{P}\,\mathrm{H}^0(\mathbb{P}^2, \mathcal{I}_{Y_0}(2))^\vee = L + |\mathcal{O}_{\mathbb{P}^2}(1)|$. For the family $\mathbb{P}^2 \times \mathbb{A}^1_t \to \mathbb{A}^1_t$ this gives an infinite order element ϕ in the birational automorphism group of the generic fiber whose specialization is the identity. Explicitly, one can choose coordinates so that $\phi\colon [x : y : z] \mapsto [x(x - ty) : (x - tz)y : (x - ty)z]$.

It is well known that birational automorphisms on K3 surfaces extend to regular automorphisms, so in the next example the specialization homomorphism is defined on Bir = Aut.

Example 2.4. Let X be a complex K3 surface of Picard rank 2 obtained as the intersection of two divisors of type $(1,1)$ and $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$. There are two projections $p_j : X \to \mathbb{P}^2$ $(j = 1,2)$, which induce involutions ι_j on X. By [25, Thm. 2.9], for a general such X it is known that the automorphism group of X is the free product $\mathrm{Aut}(X) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ generated by the involutions. On the other hand, there are special examples of such K3 surfaces where the involutions commute. In the coordinates $([X_0 : X_1 : X_2], [Y_0 : Y_1 : Y_2]) \in \mathbb{P}^2 \times \mathbb{P}^2$, one may take the complete intersection X_0 given by the equations

$$\sum_{i,j\in\{0,1\}} a_{ij} X_i Y_j = 0 \quad \text{and} \quad \sum_{i,j\in\{0,1,2\}} b_{ij} X_i^2 Y_j^2 = 0,$$

which is smooth for general coefficients a_{ij} and b_{ij}. On X_0 the covering involutions extend to (regular) involutions:

$$\iota_{1,0} : ([X_0 : X_1 : X_2], [Y_0 : Y_1 : Y_2]) \mapsto ([X_0 : X_1 : (-1) \cdot X_2], [Y_0 : Y_1 : Y_2]),$$

$$\iota_{2,0} : ([X_0 : X_1 : X_2], [Y_0 : Y_1 : Y_2]) \mapsto ([X_0 : X_1 : X_2], [Y_0 : Y_1 : (-1) \cdot Y_2]).$$

By construction, these involutions on X_0 automatically commute. This shows that the birational automorphism $\iota_1\iota_2\iota_1\iota_2$ on the general K3 surface X has infinite order but specializes to the identity on the special fiber X_0. On the special fiber, each projection p_j contracts the conic over $[0 : 0 : 1]$, so the Picard rank jumps and the covering involution does not extend to a regular involution on the family (c.f. [12, Thm. 2.1]).

One may exhibit similar behavior on K3 surfaces of type (2,2,2) in $(\mathbb{P}^1)^3$, see [24, §3] and [21, Prop. 3.5]. For an example with Enriques surfaces, see [1] (c.f. [10, IV Ex. 1.17.4]).

Example 2.5. In mixed characteristic $(0,p)$, the kernel of sp_η can contain p-torsion. It is not clear if this can be accounted for by considering an additional scheme structure on $\mathrm{Bir}(X)$. For instance, the group of p-torsion geometric points of an elliptic curve is isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ in characteristic 0, but is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ or is trivial in characteristic p, so translating by a p-torsion point that specializes to the identity gives such an example. Similarly one can construct examples by considering μ_p actions on a scheme in mixed characteristic $(0,p)$. This happens when considering μ_p-covers of schemes, and it will be an important tool in the next section.

3. Applications to birational automorphisms of Fano hypersurfaces

We now give the proofs of Theorem A and Corollary B. The key ingredients used are the specialization homomorphism for Bir, a result of the first and third authors showing that certain p-cyclic covers in characteristic p have no birational automorphisms [4, Cor. C], and a construction of Mori [18] (see also [10, V.5.14.4]) that allows us to degenerate from a hypersurface to a p-cyclic cover. We begin by recalling Mori's construction:

Construction 3.1. Let $f, g \in R[x_0, \ldots, x_{n+1}]$ be homogeneous polynomials of degree pe and e, respectively. Assume $g^p - f$ is not uniformly 0. Let $Z = (y^p - f = g - \pi y = 0) \subset \mathbb{P}_R(1^{n+2}; e)$. Then Z_η is isomorphic to the degree pe hypersurface $(g^p - \pi^p f = 0) \subset \mathbb{P}^{n+1}_K$, and Z_0 is isomorphic to a p-cyclic cover of the degree e hypersurface $(g = 0) \subset \mathbb{P}^{n+1}_k$.

There are two different degenerations that are most useful in our case:

- A p-cyclic cover in mixed characteristic $(0,p)$, and
- Mori's construction in equicharacteristic 0.

By [5, Thm. C & Ex. 1.7], these families have sustained separably uniruled modifications and sustained ruled modifications, respectively. Therefore we may apply Proposition 1.4(5).

Proposition 3.2. *Let p be a prime and let $n, e \geq 3$ be integers such that $(p-1)e \leq n-e \leq pe-3$. Furthermore, assume n is even if $p=2$. If $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ is a very general hypersurface of degree pe, then any finite order element of $\mathrm{Bir}(X)$ has order a power of p.*

Proof. The inequalities in the statement of the proposition imply that over $\overline{\mathbb{F}}_p$, a general p-cyclic cover of a degree e hypersurface in $\mathbb{P}^{n+1}$ has trivial birational automorphism group by [4, Cor. C] and is not separably uniruled by [9, Lem. 7]. So it follows from [5, Thm. C], Proposition 1.4(5), and Proposition 2.1(3) that for a very general such p-cyclic cover Z over $\mathbb{C}$, $\mathrm{Bir}_{\mathbb{C}}(Z)$ only contains elements whose orders are p-powers. By Construction 3.1, there is a family of degree pe hypersurfaces over a complex curve that degenerates to a general such p-cyclic cover. Since Z is not ruled [10, Prop. 5.12] and the total space has sustained ruled modifications [5, Ex. 1.7], we may apply Proposition 1.4(5). Together with Proposition 2.1(3) and the isomorphism between the geometric generic and very general fibers of the family [23, Lem. 2.1], this gives the result for a very general degree pe hypersurface over $\mathbb{C}$. □

Proof of Theorem A. Let $e := \lceil \frac{n+3}{p+1} \rceil$. We will first show the result for $d = pe$. By the comment after Theorem A, we may assume that $d \leq n$ (note that this implies $n \geq 3p$). The assumptions in the theorem then imply that $(p-1)e \leq n-e \leq pe-3$, so by Proposition 3.2 any torsion element in the birational automorphism group of a very general hypersurface of degree pe in $\mathbb{P}^{n+1}_{\mathbb{C}}$ has order a power of p.

For $d > pe$ we prove the result by induction, showing that the degree $d-1$ result implies the degree d result. To start, consider a pencil of hypersurfaces spanned by a smooth degree d hypersurface and a degree $d-1$ hypersurface union with a hyperplane. Assume that the union of all three is an snc divisor. Then the total space of the pencil is singular (as the dimension of the hypersurfaces is ≥ 3) and admits a small resolution by blowing up the hyperplane in the central fiber. After this blowup, the localization of the family at the reducible fiber has reduced snc central fiber with two components birational to the original ones. Thus the localized family has sustained ruled modifications by [5, Ex. 1.7].

By induction the only finite order birational automorphisms of a very general degree $d-1$ hypersurface have order a power of p. Moreover, it is not ruled by [9, Thm. 2], so we may apply Proposition 1.4(5) to the above degeneration to prove the result in degree d. □

Proof of Corollary B. Combine the results for the primes $p = 2, 3$ in Theorem A if n is even, and consider the primes $p = 3, 5$ if n is odd. □

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